Linear Real Arithmetic

$$a_1 x_1 + \cdots + a_n x_n + c \leq 0$$

Example:

$-z + 2x + 1 \leq 0$

$-2z + x + y + 1 \leq 0$

$z + x - 1 \leq 0$

$2z - 4y - 2 \leq 0$

$\boxed{x \to 0 \mid y \to 2}$

FM Rule

$$-ax + p \leq 0 \qquad bx - q \leq 0 \qquad a, b > 0$$

$$\Downarrow$$

$$bp - aq \leq 0$$

FM Rule

$$-ax + p \leq 0 \qquad bx - q \leq 0 \qquad a, b > 0$$

$$\Downarrow$$

$$bp - aq \leq 0$$

Example

$$-2z + x + y + 1 \leq 0 \qquad z + x - 1 \leq 0$$

$$\Downarrow$$

$$x + y + 1 + 2x - 2 \leq 0$$

$$\Downarrow$$

$$3x + y - 1 \leq 0 \qquad x \to 0, \; y \to 2$$

Nonlinear Real Arithmetic

Polynomials

Univariate Polynomials

Multivariate Polynomials

Polynomial division (univariate)

Given f, g find q, r

$$f = q g + r \qquad \deg(r) < \deg(g)$$

Example:

$$3x^3 + x^2 + 1 = (3x+1)(x^2+1) - 3x$$

Polynomial Division Algorithm

$q := 0$

$r := f$

while $\deg(r) \geq \deg(g)$

INVARIANT $f = q\,g + r$

$d := \deg(r) - \deg(g)$

$q := q + \dfrac{LC(r)}{LC(g)} \cdot x^d$

$r := r - \dfrac{LC(r)}{LC(g)} \cdot x^d \cdot g$

LC leading coefficient

Polynomial division Example

$f: 3x^3 + x^2 + 1$

$g: x^2 + 1$

$q := 0$ $\quad\quad$ $LC(g) = 1$ $\quad\quad$ $d = 3-2 = 1$

$r := 3x^3 + x^2 + 1$ $\quad\quad$ $LC(r) = 3$

Polynomial division Example

$f: 3x^3 + x^2 + 1$

$g: x^2 + 1$

$q := 0$ $\qquad LC(g) = 1 \qquad d = 3 - 2 = 1$

$r := 3x^3 + x^2 + 1$ $\qquad LC(r) = 3$

$q := 3x$ $\qquad LC(r) = 1$

$r := 3x^3 + x^2 + 1 - 3x(x^2 + 1) = x^2 - 3x + 1$ $\qquad d = 2 - 2 = 0$

$q := 3x + 1$

$r := x^2 - 3x + 1 - 1(x^2 + 1) = -3x$

$f = qg + r$

if $g(\gamma) = 0$

Then $f(\gamma) = r(\gamma)$

Polynomial Sequence

$$S = \langle P_0, P_1, \ldots, P_m \rangle$$

$Var(S, a)$ Number of sign variations at a

Polynomial Sequence

$S = \langle P_0, P_1, \ldots, P_m \rangle$

$Var(S, a)$ Number of sign variations at a

Example:

$S = \langle 3x^4 - 3x^2 - 2,\ 12x^3 - 6x,\ x^2 + 1,\ x - 1,\ -1 \rangle$

at 1

Ignore zeros

$\langle -2, 6, 2, 0, -1 \rangle$

$Var(S, 1) = 2$

Sturm Sequence for (f, g)

$h_0 = f$

$h_1 = g$

$h_0 = q_1 h_1 - h_2$

$h_1 = q_2 h_2 - h_3$

$\vdots$

$h_{i-1} = q_i h_i - h_{i+1}$

$\vdots$

$h_{n-1} = q_n h_n$

Sturm Sequence for (f,g)

$$h_0 = f$$
$$h_1 = g$$
$$h_0 = q_1 h_1 - h_2$$
$$h_1 = q_2 h_2 - h_3$$
$$\vdots$$
$$h_{i-1} = q_i h_i - h_{i+1}$$
$$\vdots$$
$$h_{m-1} = q_m h_m$$

$$h_0 = f$$
$$h_1 = g$$
$$h_2 = -\mathrm{REM}(h_0, h_1)$$
$$h_3 = -\mathrm{REM}(h_1, h_2)$$
$$\vdots$$
$$h_{i+1} = -\mathrm{REM}(h_{i-1}, h_i)$$
$$\vdots$$
$$h_m = -\mathrm{REM}(h_{m-2}, h_{m-1})$$
$$\mathrm{REM}(h_{m-1}, h_m) = 0$$

Sturm Sequence for (f, g)

$h_0 = f$

$h_1 = g$

$h_0 = q_1 h_1 - h_2$

$h_1 = q_2 h_2 - h_3$

$\vdots$

$h_{i-1} = q_i h_i - h_{i+1}$

$\vdots$

$h_{n-1} = q_n h_n$

$h_0(\gamma) = h_1(\gamma) = 0$

$\Rightarrow$

$h_i(\gamma) = 0 \quad i \in [0, n]$

$h_n(\gamma) = 0$

$\Rightarrow$

$h_i(\gamma) = 0 \quad i \in [0, n]$

$h_j(\gamma) = h_{j+1}(\gamma) = 0$

$\Rightarrow$

$h_i(\gamma) = 0 \quad i \in [0, n]$

Sturm Theorem

$S = \text{Sturm}(f, f')$ $\quad$ f' is the derivative of f

$a < b$

$f(a) \neq 0, \quad f(b) \neq 0$

$$\Rightarrow \text{Var}(S, a) - \text{Var}(S, b) = \#\{ \gamma \mid a < \gamma < b, f(\gamma) = 0 \}$$

Number of roots in the interval (a, b)

Example

$$h_0 = f = x^4 - 10x^3 + 32x^2 - 38x + 15$$

$$h_1 = f' = 4x^3 - 30x^2 + 64x - 38$$

$$h_2 = -\mathrm{REM}(h_0, h_1) = \frac{11}{4}x^2 - \frac{23}{2}x + \frac{35}{4} \rightsquigarrow 11x^2 - 46x + 35$$

$$h_3 = \frac{512}{121}x - \frac{512}{121} \rightsquigarrow x - 1$$

Example

$$h_0 = f = x^4 - 10x^3 + 32x^2 - 38x + 15 \qquad (x-1)^2(x-3)(x-5)$$

$$h_1 = f' = 4x^3 - 30x^2 + 64x - 38$$

$$h_2 = -\mathrm{REM}(h_0, h_1) = \frac{11}{4}x^2 - \frac{23}{2}x + \frac{35}{4} \rightsquigarrow 11x^2 - 46x + 35$$

$$h_3 = \frac{512}{121}x - \frac{512}{121} \rightsquigarrow x - 1$$

	0	2	4	6
h_0	+	+	−	+
h_1	−	+	−	+
h_2	+	−	+	+
h_3	−	+	+	+

We can decide formulas such as

$$p > 0 \qquad p = 0 \qquad p < 0$$

Example $x^2 + 1 < 0$

	$-\infty$	∞
x^2+1	$+$	$+$
$2x$	$-$	$+$
-1	$-$	$-$

We can decide formulas such as

$$p > 0 \qquad p = 0 \qquad p < 0$$

Example $x^2 + 1 < 0$

	$-\infty$	∞
x^2+1	$+$	$+$
$2x$	$-$	$+$
-1	$-$	$-$

Only the sign of the leading coefficient matters at $-\infty$ and $+\infty$

Sturm-Tarski theorem

$S = \mathrm{Sturm}(f, f'g)$

$a < b$

$f(a) \neq 0 \quad f(b) \neq 0$

$\Rightarrow$

$$\mathrm{Var}(S, a) - \mathrm{Var}(S, b) = \#\{\gamma \mid a < \gamma < b,\ f(\gamma) = 0,\ g(\gamma) > 0\}$$

$$- \#\{\gamma \mid a < \gamma < b,\ f(\gamma) = 0,\ g(\gamma) < 0\}$$

Sturm-Tarski theorem

$S = \mathrm{Sturm}(f, f'g)$

$a < b$

$f(a) \neq 0 \quad f(b) \neq 0$

$\Rightarrow$

$$\underbrace{\mathrm{Var}(S,a) - \mathrm{Var}(S,b)}_{\mathrm{TaQ}(g,f;(a,b))} = \#\{x \mid a < x < b,\ f(x) = 0,\ g(x) > 0\} - \#\{x \mid a < x < b,\ f(x) = 0,\ g(x) < 0\}$$

$$\mathrm{TaQ}(g,f) = \mathrm{TaQ}(g,f;(-\infty,\infty))$$

$TaQ(1, f)$ number of roots of f

$TaQ(g, f)$ $\underbrace{\#(g>0)}_{\#\{x \mid f(x)=0,\ g(x)>0\}} - \#(g<0)$

$\#\{x \mid f(x), g(x)<0\}$

$TaQ(g^2, f) \rightsquigarrow ?$

$\mathrm{TaQ}(1, f)$ = number of roots of f

$$\mathrm{TaQ}(g, f) = \underbrace{\#(g>0)}_{\#\{x \mid f(x)=0,\ g(x)>0\}} - \#(g<0)$$

$$\#\{x \mid f(x),\ g(x)<0\}$$

$$\mathrm{TaQ}(g^2, f) = \#(g>0) + \#(g<0)$$

$$\#(g=0) + \#(g>0) + \#(g<0) = \mathrm{TaQ}(1,f)$$

$$\#(g>0) - \#(g<0) = \mathrm{TaQ}(g,f)$$

$$\#(g>0) + \#(g<0) = \mathrm{TaQ}(g^2,f)$$

$$\#(g=0) + \#(g>0) + \#(g<0) = TaQ(1, f)$$

$$\#(g>0) - \#(g<0) = TaQ(g, f)$$

$$\#(g>0) + \#(g<0) = TaQ(g^2, f)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \#(g=0) \\ \#(g>0) \\ \#(g<0) \end{pmatrix} = \begin{pmatrix} TaQ(1, f) \\ TaQ(g, f) \\ TaQ(g^2, f) \end{pmatrix}$$

We can decide formulas such as

$$P = 0 \wedge g < 0$$

$$P = 0 \wedge g > 0$$

$$P = 0 \wedge g = 0$$

Example:

$$\underbrace{x^2 - 1}^{f} = 0 \wedge \underbrace{x + 1}^{g} > 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \#(g=0) \\ \#(g>0) \\ \#(g<0) \end{pmatrix} = \begin{pmatrix} TaQ(1,f) \\ TaQ(g,f) \\ TaQ(g^2,f) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

We can decide formulas such as

$$P=0 \wedge g<0$$

$$P=0 \wedge g>0$$

$$P=0 \wedge g=0$$

Example:

$$\underbrace{x^2-1}_{f}=0 \wedge \underbrace{x+1}_{g}>0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \#(g=0) \\ \#(g>0) \\ \#(g<0) \end{pmatrix} = \begin{pmatrix} TaQ(1,f) \\ TaQ(g,f) \\ TaQ(g^2,f) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\#(g=0)=1$$

$$\#(g>0)=1$$

$$\#(g<0)=0$$

We can decide formulas such as

$$p=0 \wedge g<0$$

$$p=0 \wedge g>0$$

$$p=0 \wedge g=0$$

Is $x^2-1=0 \wedge x+1<0$ satisfiable?

Example:

$$\underbrace{x^2-1}_{f}=0 \wedge \underbrace{x+1}_{g}>0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \#(g=0) \\ \#(g>0) \\ \#(g<0) \end{pmatrix} = \begin{pmatrix} TaQ(1,f) \\ TaQ(g,f) \\ TaQ(g^2,f) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$\#(g=0)=1$

$\#(g>0)=1$

$\#(g<0)=0$

$$\mathrm{TaQ}(g_1 g_2, f) = \#\{\gamma \mid f(\gamma)=0,\ g_1(\gamma)g_2(\gamma)>0\}$$
$$- \#\{\gamma \mid f(\gamma)=0,\ g_1(\gamma)g_2(\gamma)<0\}$$

$$\mathrm{TaQ}(g_1 g_2, f) = \#\{\gamma \mid f(\gamma)=0,\ g_1(\gamma)g_2(\gamma)>0\}$$
$$- \#\{\gamma \mid f(\gamma)=0,\ g_1(\gamma)g_2(\gamma)<0\}$$
$$= \#(g_1>0, g_2>0) + \#(g_1<0, g_2<0)$$
$$- \#(g_1>0, g_2<0) - \#(g_1<0, g_2>0)$$

$$\mathrm{TaQ}(g_1^2 g_2, f) = \#(g_1 > 0, g_2 > 0) + \#(g_1 < 0, g_2 > 0) + - \#(g_1 > 0, g_2 < 0) + - \#(g_1 < 0, g_2 < 0)$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}
\begin{matrix}
\#(g_1=0,\, g_2=0) \\
\#(g_1>0,\, g_2=0) \\
\#(g_1<0,\, g_2=0) \\
\#(g_1=0,\, g_2>0) \\
\#(g_1>0,\, g_2>0) \\
\#(g_1<0,\, g_2>0) \\
\#(g_1=0,\, g_2<0) \\
\#(g_1>0,\, g_2<0) \\
\#(g_1<0,\, g_2<0)
\end{matrix}
=
\begin{matrix}
TaQ(1,f) \\
TaQ(g_1,f) \\
TaQ(g_1^2,f) \\
TaQ(g_2,f) \\
TaQ(g_1 g_2,f) \\
TaQ(g_1^2 g_2,f) \\
TaQ(g_2^2,f) \\
TaQ(g_1 g_2^2,f) \\
TaQ(g_1^2 g_2^2,f)
\end{matrix}
$$

$$
\left(\begin{array}{ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}\right)
\begin{pmatrix}
\#(g_1=0,\ g_2=0) \\
\#(g_1>0,\ g_2=0) \\
\#(g_1<0,\ g_2=0) \\
\#(g_1=0,\ g_2>0) \\
\#(g_1>0,\ g_2>0) \\
\#(g_1<0,\ g_2>0) \\
\#(g_1=0,\ g_2<0) \\
\#(g_1>0,\ g_2<0) \\
\#(g_1<0,\ g_2<0)
\end{pmatrix}
=
\begin{pmatrix}
TaQ(1,f) \\
TaQ(g_1,f) \\
TaQ(g_1^2,f) \\
TaQ(g_2,f) \\
TaQ(g_1g_2,f) \\
TaQ(g_1^2g_2,f) \\
TaQ(g_2^2,f) \\
TaQ(g_1g_2^2,f) \\
TaQ(g_1^2g_2^2,f)
\end{pmatrix}
$$

$$
\left(\begin{array}{ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}\right)
\begin{pmatrix}
\#(g_1=0,\ g_2=0) \\
\#(g_1>0,\ g_2=0) \\
\#(g_1<0,\ g_2=0) \\
\#(g_1=0,\ g_2>0) \\
\#(g_1>0,\ g_2>0) \\
\#(g_1<0,\ g_2>0) \\
\#(g_1=0,\ g_2<0) \\
\#(g_1>0,\ g_2<0) \\
\#(g_1<0,\ g_2<0)
\end{pmatrix}
=
\begin{pmatrix}
TaQ(1,f) \\
TaQ(g_1,f) \\
TaQ(g_1^2,f) \\
TaQ(g_2,f) \\
TaQ(g_1g_2,f) \\
TaQ(g_1^2g_2,f) \\
TaQ(g_2^2,f) \\
TaQ(g_1g_2^2,f) \\
TaQ(g_1^2g_2^2,f)
\end{pmatrix}
$$

We can decide formulas such as

$$f=0 \wedge g_1>0 \wedge g_2<0$$

$$f=0 \wedge g_1<0 \wedge g_2=0$$

We can generalize to $f, \{g_1, \ldots, g_k\}$

3^k equations!

We can do better!

Ore, Kozen, Reif optimization.

- Number of roots of $f \ll 3^k$
- Each "unknown" ≥ 0

We can do better!

Ore, Kozen, Reif optimization.

- Number of roots of $f \ll 3^k$
- Each "unknown" ≥ 0

Solve the system incrementally

$f, \{g_1\} \longrightarrow \#(g_1 > 0) = 0 \longrightarrow \#(g_1 > 0, *) = 0$

$f, \{g_1, g_2\}$ found "3^{k-1} zeros"

...

$f, \{g_1, \ldots, g_k\}$

What about formulas such as

$$g_1 > 0 \wedge g_2 < 0$$

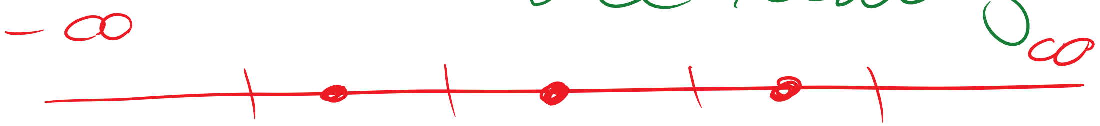
Given $\{g_1, \dots, g_k\}$

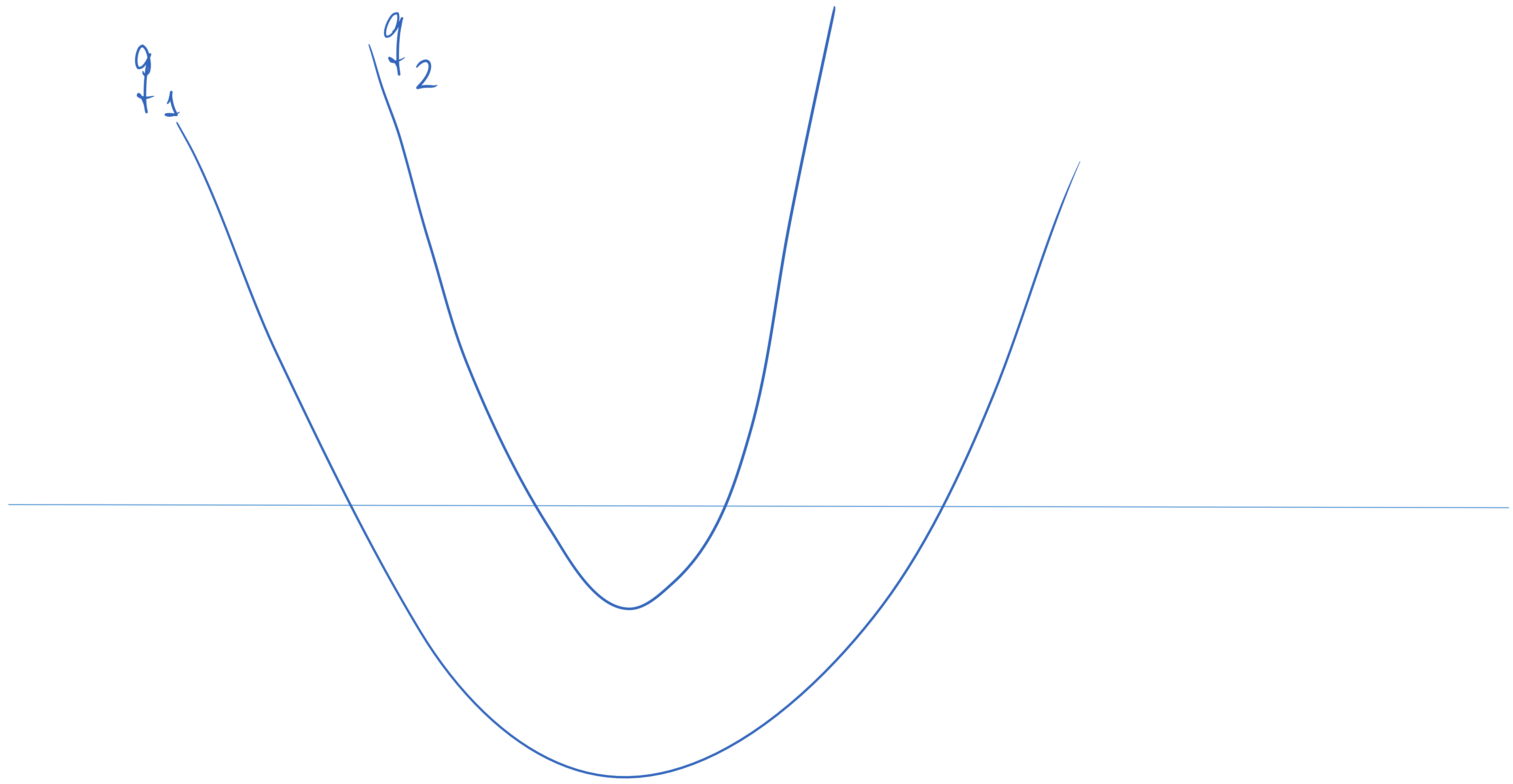
Make $f = g_1 \cdot g_2 \cdot \dots \cdot g_k$

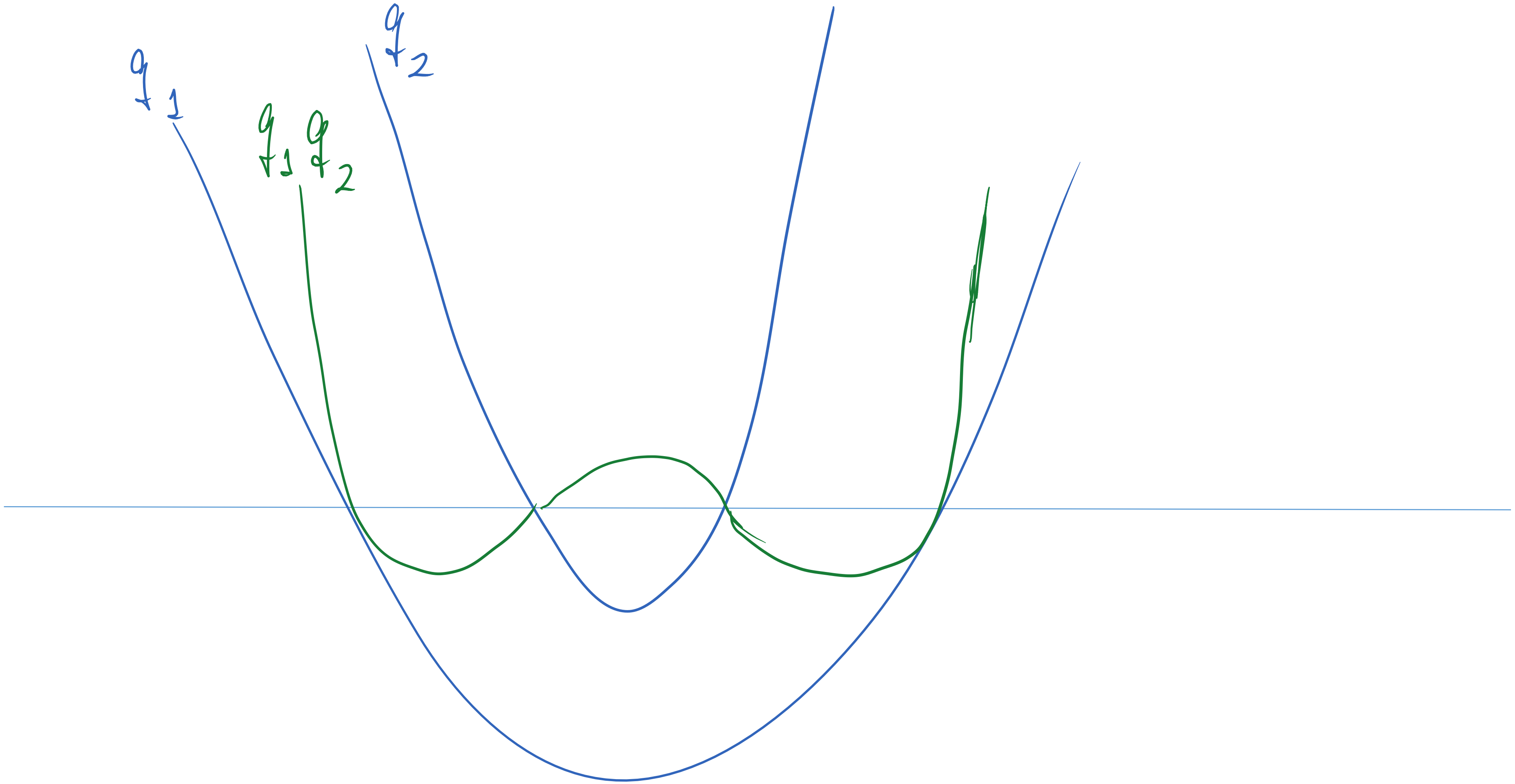
1) $TaQ(1, f) = 0 \Rightarrow$ g_i's have constant sign
use sign of leading coefficient.

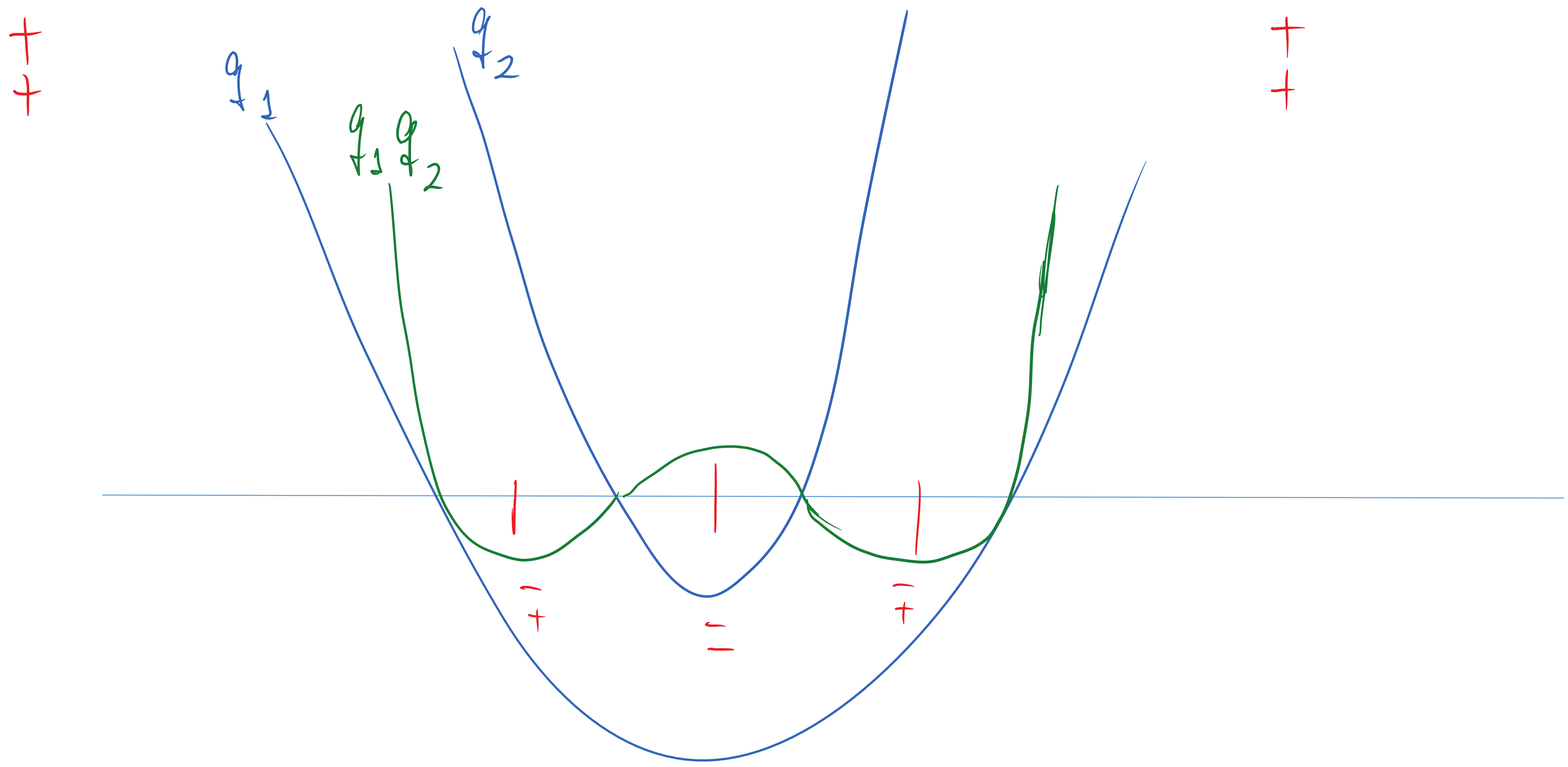
2) $TaQ(1, f) = 1 \Rightarrow$ g_i's have at most one root
use Leading coefficient to compute
sign at $-\infty$ and $+\infty$

3) $TaQ(1, f) > 1 \Rightarrow$ $-\infty$, ∞ and $f' = 0$ contains
all realizable sign conditions.




g_1
g_2


g_1
g_2
$g_1 g_2$



+
+
q_1
q_2
$q_1 q_2$
+
+
$-$
+
$-$
$-$
$-$
+

Multivariate case

$$- \quad y^2z^2 + z^2 + xyz + z + x^3 + y^2$$

$$\Downarrow$$

$$\underbrace{(y^2+1)}_{a_2}z^2 + \underbrace{(xy+1)}_{a_1}z + \underbrace{(x^3+y^2)}_{a_0} \qquad \mathbb{Q}[x,y][z]$$

Multivariate case

- $y^2 z^2 + z^2 + xyz + z + x^3 + y^2$

$$\Downarrow$$

$$\underbrace{(y^2+1)}_{a_2} z^2 + \underbrace{(xy+1)}_{a_1} z + \underbrace{(x^3+y^2)}_{a_0} \qquad \mathbb{Q}[x,y][z]$$

- Sign of the leading coefficients

Pseudo Polynomial Division

$$\underbrace{\mathbb{Q}[x,y]}[z]$$

$\hookrightarrow$ This is not a field (No multiplicative inverse)

$$\underbrace{LC(g)^k} f = q\,g + r$$

$\nwarrow$ Trick for cleaning denominators

Example:

$f: z^2+1 \quad g: xz+1$

$q=0$

$r=z^2+1$

$l=1$

Example:

$f: z^2+1 \quad g: xz+1$

$q = 0$

$r = z^2+1$

$l = 1$

$q = z$

$r = x(z^2+1) - z(xz+1) = -z + x$

$l = x$

Example:

$f: z^2+1 \quad g: xz+1$

$q=0$

$r=z^2+1$

$l=1$

$q=z$

$r=x(z^2+1)-z(xz+1)=-z+x$

$l=x$

$q=xz-1$

$r=x(-z+x)-(-1)(xz+1)=x^2+1$

$l=x^2$

Example:

$f: z^2+1 \qquad g: xz+1$

$$x^2(z^2+1) = (xz-1)(xz+1) + (x^2+1)$$

$q = 0$

$r = z^2+1$

$l = 1$

$q = z$

$r = x(z^2+1) - z(xz+1) = -z+x$

$l = x$

$q = xz-1$

$r = x(-z+x) - (-1)(xz+1) = x^2+1$

$l = x^2$

$\ell := 1$

$q := 0$

$r := f$

while $\deg(r) \geq \deg(g)$

INVARIANT $\ell . f = q\, g + r$

$\ell := LC(g) . \ell$; $q := LC(g) . q$; $r := LC(g) . r$

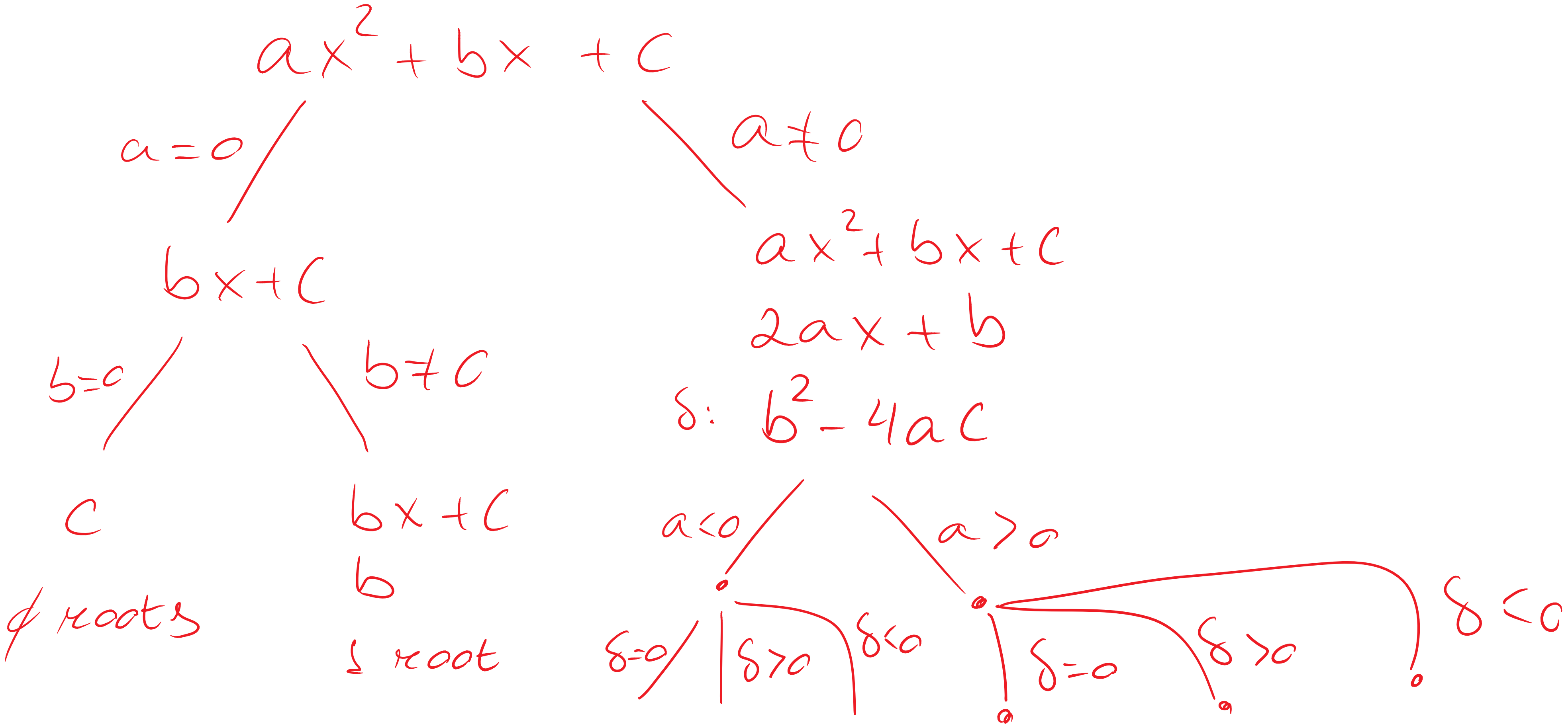
$d := \deg(r) - \deg(g)$

$$q := q + \frac{LC(r)}{LC(g)} . x^d$$

$$r := r - \frac{LC(r)}{LC(g)} . x^d . g$$

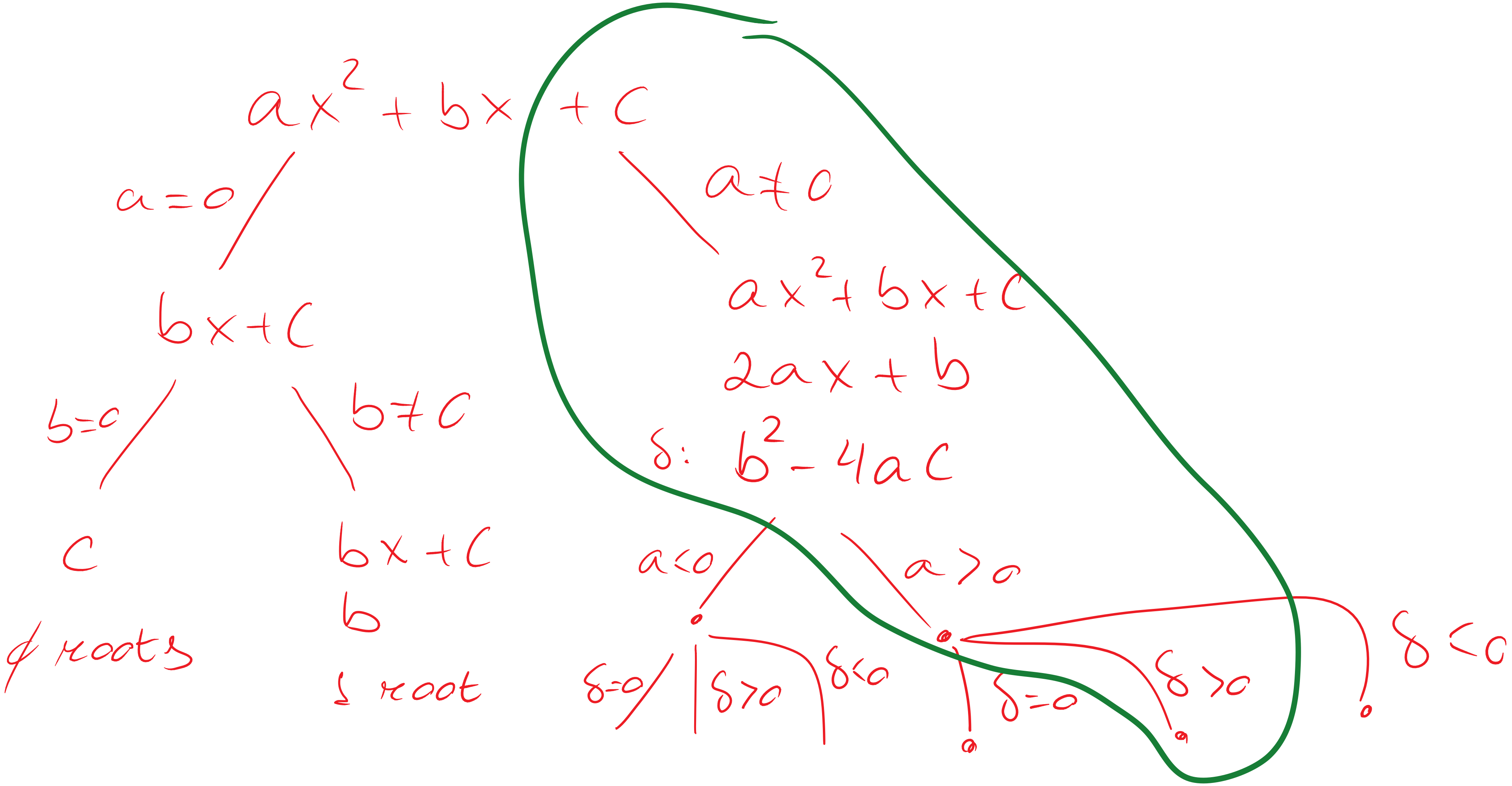
Sturm Tree

– "Branch on the sign of the leading coeff."



Sturm Tree

– "Branch on the sign of the leading coeff."



$ax^2 + bx + c$

$2ax + b$

$b^2 - 4ac$

Assumptions.

$a > 0$

$b^2 - 4ac > 0$

$ax^2 + bx + c$

$2ax + b$

$b^2 - 4ac$

Assumptions.

$a > 0$

$b^2 - 4ac > 0$

$-\infty$	∞
$+$	$+$
$-$	$+$
$+$	$+$

$ax^2 + bx + c$

$2ax + b$

$b^2 - 4ac$

Assumptions.

$a > 0$

$b^2 - 4ac = 0$

$-\infty$ ∞

$+$ $+$

$-$ $+$

0 0

$ax^2 + bx + c$

$2ax + b$

$b^2 - 4ac$

Assumptions.

$a > 0$

$b^2 - 4ac < 0$

$-\infty$

$+$

$-$

$-$

∞

$+$

$+$

$-$

- Given a polynomials $\{g_1, \ldots, g_n\}$ in $\mathbb{Q}[\vec{Y}][X]$
- Given an assignment for $\vec{Y}$
- We have to consider only one branch of the Sturm Tree.

- Given a polynomials $\{g_1, \ldots, g_n\}$ in $\mathbb{Q}[\vec{y}][X]$
- Given an assignment for $\vec{y}$
- We have to consider only one branch of the Sturm Tree.

$$ax^2 + bx + c \qquad a \to 1,\ b \to 2,\ c \to 1$$

$$2ax + b \qquad a = 1 > 0$$

$$b^2 - 4ac \qquad b^2 - 4ac = 0$$

Example: $y > 0 \wedge (y+2)x^4 + (y^2+1)x^2 + 1 < 0$

$y \to 1$

$$3x^4 + 2x^2 + 1 < 0 \to 0 \text{ roots}$$

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$\Downarrow$

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1

$-\infty$	∞
+	+
−	+
−	−
+	+
+	+

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1

$$\boxed{\begin{gathered}(y+2) > 0 \\ (y^2+1) > 0 \\ (-y^4 - 2y^2 + 2y + 3) > 0\end{gathered}}$$

$$\Downarrow$$

$(y+2)x^4 + (y^2+1)x^2 + 1 > 0$

$-\infty$	∞
+	+
−	+
−	−
+	+
+	+

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1

$$\boxed{\begin{array}{l} (y+2) > 0 \\ \text{redundant} \\ \cancel{(y^2+1) > 0} \\ (-y^4-2y^2+2y+3) > 0 \end{array}}$$

$$\Downarrow$$

$(y+2)x^4 + (y^2+1)x^2 + 1 > 0$

$-\infty$	∞
+	+
−	+
−	−
+	+
+	+

Example: $y > 0 \wedge (y+2)x^4 + (y^2+1)x^2 + 1 < 0$

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$(-y^4 - 2y^2 + 2y + 3)x$

1

$\overbrace{(-y^4 - 2y^2 + 2y + 3)}^{\delta} > 0$

$-\infty$	$\delta > 0$	∞	$-\infty$	$\delta = 0$	$+\infty$	$-\infty$	$\delta > 0$	∞
+		+	+		+	+		+
−		+	−		+	−		+
−		−	−		−	−		−
+		+	0		0	−		−
+		+				+		+

Example: $y>0 \wedge (y+2)x^4 + (y^2+1)x^2 + 1 < 0$

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$-(y^2+1)x^2 - 1$

$(-y^4 - 2y^2 + 2y + 3)x$

1

redundant

δ

$\cancel{(-y^4 - 2y^2 + 2y + 3) > 0}$

$-\infty$	$\delta>0$	∞	$-\infty$	$\delta=0$	$+\infty$	$-\infty$	$\delta>0$	∞
+		+	+		+	+		+
−		+	−		+	−		+
−		−	−		−	−		−
+		+	0		0	−		−
+		+				+		+

Example: $y > 0 \wedge (y+2)x^4 + (y^2+1)x^2 + 1 < 0$

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$(y+2)x^4 + (y^2+1)x^2 + 1$

$4(y+2)x^3 + 2(y^2+1)x$

$-(y^2+1)x^2 - 1$

$(-y^4 - 2y^2 + 2y + 3)x$

1

$(y+2) > 0$

$$\Downarrow$$

$\neg((y+2)x^4 + (y^2+1)x^2 + 1 < 0)$

Example: $y > 0 \wedge \underline{(y+2)x^4 + (y^2+1)x^2 + 1 < 0}$

$y \to 1$

$$\Downarrow$$

$3x^4 + 2x^2 + 1 < 0 \to$ 0 roots

$$\neg(y+2>0) \vee \underline{\neg((y+2)x^4 + (y^2+1)x^2 + 1 < 0)}$$

resolve

$$\Downarrow$$

$$\neg(y+2>0)$$

$$\Downarrow$$

$$y \leq -2$$